

Quantization of a Constant of Motion for the Harmonic Oscillator with a Time-Explicitly Depending Force

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ABSTRACT

The quantization of a constant of motion for the harmonic oscillator with a time-explicitly depending external force is carried out. This quantization approach is compared with the normal Hamiltonian quantization approach. Numerical results show that there are qualitative and quantitative differences for both approaches, suggesting that the quantization of this constant of motion may be verified experimentally.

I. Introduction

Modern Physics has emerged as theories formulated in terms of Hamiltonians or Lagrangians structures. Among them, one could mention classical physics, field theories [1], quantum mechanics [2] and statistical mechanics [3]. Nature has been kind enough to allows its description , so far, in terms of Hamiltonian or Lagrangian formulation, despite of some mathematical problems that these formulations have by themselves [4]. These problems appear mainly for dissipative [5] or time-explicitly depending phenomena [6], where one wonder whether the associated constant of motion or Hamiltonian is the relevant parameter to study the associated physics. For the harmonic oscillator with a time-explicitly depending force (HOTED), it is possible to know a constant of motion which is also time-explicitly depending [7]. Therefore, one may wonder about any difference that HOTED could have from the quantum mechanics point of view. In this paper, the quantization of a constant of motion fort HOTED is associated and compared with the usual Hamiltonian approach.

II. Constant of Motion and Quantization Approach

Consider that the harmonic oscillator is perturbed by the time-explicitly depending force given by

$$f(t) = A \sin \Omega t , \quad (1)$$

where A and Ω are the amplitude and the frequency of the force. The resulting classical dynamical system can be written as

$$\frac{dx}{dt} = v \quad (2a)$$

and

$$\frac{dv}{dt} = -\omega^2 x + \frac{A}{m} \sin \Omega t , \quad (2b)$$

where ω is the frequency of the free harmonic oscillations. A constant of motion associated to [2] for $\Omega \neq \omega$ is given by [7] the function

$$K(x, v, t) = \frac{m}{2} (v^2 + \omega^2 x^2) + \frac{A}{\Omega^2 - \omega^2} \left(\Omega v \cos \Omega t + \omega^2 x \sin \Omega t \right) - \frac{A^2}{2m(\Omega^2 - \omega^2)} \sin^2(\Omega t) . \quad (3)$$

One can readily see that (3) is a constant of motion since it satisfies the following partial differential equation

$$v \frac{\partial K}{\partial x} + [-\omega^2 x + \frac{A}{m} \sin \Omega t] \frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0 . \quad (4)$$

The approach used here to quantize (3) is based on the construction of the associated Shrödinger's equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \widehat{K}(\hat{x}, \hat{v}, t) \Psi , \quad (5)$$

where Ψ is the wave function, $\Psi = \Psi(x, t)$, and \widehat{K} is the hermitian operator associated to function (3). This hermitian operator is constructed by making the following substitutions

$$x \longrightarrow \hat{x} = x \quad \text{and} \quad v \longrightarrow \hat{v} = -i\frac{\hbar}{m} \frac{\partial}{\partial x} \quad (6)$$

which bring about the Shrödinger's equation

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} = & -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \Psi \\ & + \left\{ -\frac{A^2}{2m(\Omega^2 - \omega^2)} \sin^2 \Omega t + \frac{A\omega^2 x}{\Omega^2 - \omega^2} \sin \Omega t \right\} \Psi \\ & - i \frac{A\Omega \hbar \cos \Omega t}{m(\Omega^2 - \omega^2)} \frac{\partial \Psi}{\partial x} . \end{aligned} \quad (7)$$

The first line on Eq. (7) represents the pure harmonic quantum oscillator. The eigenfunctions of the pure harmonic oscillator are well known [8] and are better handle with annihilation and creation operators, \tilde{a} and \tilde{a}^+ . \hat{x} and \hat{v} are written in terms of these operators as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\tilde{a} + \tilde{a}^+) , \quad (8a)$$

and

$$\hat{v} = -i\sqrt{\frac{\hbar\omega}{2m}} (\tilde{a} - \tilde{a}^+) . \quad (8b)$$

Then, Eq. (3) can be written as

$$\begin{aligned} \widehat{K} = & \widehat{K}_o(\tilde{a}^+, \tilde{a}) - \frac{A^2}{2m(\Omega^2 - \omega^2)} \sin^2 \Omega t \\ & + \frac{A}{\Omega^2 - \omega^2} \sqrt{\frac{\hbar\omega}{2m}} \left\{ [\omega \sin \Omega t - i\Omega \cos \Omega t] \tilde{a} + [\omega \sin \Omega t + i\Omega \cos \Omega t] \tilde{a}^+ \right\} , \end{aligned} \quad (9a)$$

where $\hat{K}_o(\tilde{a}^+, \tilde{a})$ is given by

$$\hat{K}_o = \hbar\omega(\tilde{a}^+\tilde{a} + 1/2) . \quad (9b)$$

If $|n\rangle$ represents an eigenfunction of the pure harmonic oscillator, \hat{K}_o , one has the usual properties

$$\tilde{a}|n\rangle = \sqrt{n} |n-1\rangle , \quad (10a)$$

$$\tilde{a}^+|n\rangle = \sqrt{n+1} |n+1\rangle , \quad (10b)$$

$$\tilde{a}^+\tilde{a}|n\rangle = n|n\rangle \quad (10c)$$

and

$$[\tilde{a}, \tilde{a}^+] = 1 , \quad (10c)$$

where symbol $[,]$ represents the commutator between operators, $[\tilde{a}, \tilde{a}^+] = \tilde{a}\tilde{a}^+ - \tilde{a}^+\tilde{a}$.

Now, proposing in (5) a solution of the form

$$\Psi(x, t) = \sum_{n=0}^{\infty} c_n(t) |n\rangle , \quad (11)$$

one gets the system of equations

$$\begin{aligned} i\hbar\dot{c}_m = & \left[\hbar\omega(n+1/2) - \frac{A^2}{2m(\Omega^2 - \omega^2)} \sin^2 \Omega t \right] c_m \\ & + \frac{A}{\Omega^2 - \omega^2} \sqrt{\frac{\hbar\omega}{2m}} \left\{ \sqrt{m+1}(\omega \sin \Omega t - i\Omega \cos \Omega t) c_{m+1} + \sqrt{m}(\omega \sin \Omega t + i\Omega \cos \Omega t) c_{m-1} \right\} \end{aligned} \quad (12)$$

which can be written in terms of the parameters

$$\tau = \omega t , \quad \epsilon = A/\hbar\Omega , \quad \bar{\hbar} = \hbar/m\omega \quad \text{and} \quad \rho = \Omega/\omega \quad (13)$$

as

$$\begin{aligned} ic'_m = & \left(n + 1/2 - \frac{\epsilon^2}{2\bar{\hbar}(1 - \rho^2)} \sin^2 \rho\tau \right) c_m \\ & + \epsilon\sqrt{2\bar{\hbar}} (1 - \rho) \left\{ \sqrt{m+1}(\sin \rho\tau - i\rho \cos \rho\tau) c_{m+1} + \sqrt{m}(\sin \rho\tau + i\rho \cos \rho\tau) c_{m-1} \right\} , \end{aligned} \quad (14)$$

where c'_m denotes the expression $c'_m = dc_m/d\tau$.

On the other hand, the Hamiltonian quantization approach for the system (2) is formulated by the Shrödinger's equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi , \quad (15)$$

where the Hamiltonian operator \hat{H} is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2 - xA \sin \Omega t . \quad (16)$$

The operators \hat{x} and \hat{p} can be written in terms of the annihilation and creation operators, a and a^+ , as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^+) \quad (17a)$$

and

$$\hat{p} = -i\sqrt{\frac{\hbar\omega m}{2}} (a - a^+) , \quad (17b)$$

where a and a^+ satisfy the same relations (10). Written (16) in terms of these operators, it follows that

$$\hat{H} = \hat{H}_o(a^+, a) - \sqrt{\frac{\hbar}{2m\omega}} A \sin \Omega t (a + a^+) , \quad (18)$$

where $\hat{H}_o(a^+, a)$ is given by $\hat{H}_o = \hbar\omega(a^+a + 1/2)$. Using the same expansion (10) and parameters (13), one gets the system of equations

$$ic'_m = (m + 1/2)c_m - \sqrt{\frac{\hbar}{2}} \epsilon \rho \sin \rho \tau (\sqrt{m} c_{m-1} + \sqrt{m+1} c_{m+1}) . \quad (19)$$

III. Numerical Solution

To know the evolution of $c_m(\tau)$ for Eqs. (14) and (15), these can be solve numerically. The fixed parameters used are $\bar{\hbar} = 0.4$ and $\rho = 6.25$, and one studies the dependence of the system on the parameters ϵ and τ . The initial conditions are

$$c_o(0) = 1, \quad , c_j(0) = 0 \quad \text{for } j = 1, 2, \dots \quad (20)$$

The Fig. 1 shows the evolution of the probabilities $|c_o(\tau)|^2$ and $|c_1(\tau)|^2$ and their dependence on the parameter ϵ . The upper plots correspond to the quantization of the constant of motion (Eq. 14), and the lower plots correspond to the quantization of the Hamiltonian (Eq. 19). For the same ϵ , the peak values of the probabilities $|c_o(\tau)|^2$ and $|c_1(\tau)|^2$ occur at different times (τ) and with different amplitudes.

The Fig. 2 shows the maximum number of exited states involved in the dynamics of the system for a given ϵ . The criterion used is the following: one defines that a state

$|n\rangle$ is involved in the dynamics if the probability that the system to be in this state is higher than 0.0001, $|c_n(\tau)|^2 > 0.0001$. As one can see, for a given ϵ the number of levels involved in the Hamiltonian quantization can be much larger than the levels involved in the constant of motion quantization.

The Fig. 3 shows the evolution of the expected value of x^2 ,

$$\langle x^2 \rangle = \langle \Psi | x^2 | \Psi \rangle = \sum_{m,n=0}^{\infty} c_m^*(\tau) c_n(\tau) \langle m | x^2 | n \rangle, \quad (21)$$

for the constant of motion and Hamiltonian quantization. As one can see, the value for the Hamiltonian case is about one order of magnitude higher than its value for the constant of motion case.

III. Conclusions

The quantization of the harmonic oscillator with a time-explicitly depending external force has been studied from the point of view of the constant of motion. The results have been compared with the known case of Hamiltonian approach. The probabilities $|c_o(\tau)|^2$ and $|c_1(\tau)|^2$, the expected value $\langle x^2 \rangle$ and the number of excited states involved in the dynamics are quite different for the constant of motion and Hamiltonian approaches. These results may be checked experimentally taking care that the external force added to the harmonic oscillator system is of the form given by Eq. (1).

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Figure Captions

Fig.1 Evolution of the probabilities $|c_o(\tau)|^2$ and $|c_1(\tau)|^2$ for the system (14) and (19). The value of the other parameters are $\bar{\hbar} = 0.4$ and $\rho = 6.25$. The initial conditions are given by Eq. (20).

Fig. 2 Number of exited states involved in the dynamics as a function of the parameter ϵ . The other parameters are the same as Fig. 1.

Fig. 3 Evolution of the expected value $\langle x^2 \rangle$ for the constant of motion and Hamiltonian quantization approaches. The parameter epsilon takes the values $\epsilon = 0, 5, 10$. The other parameters and initial conditions are the same as Fig. 1.